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Absence of Phase Transitions in Certain One-Dimensional Long-Range Random Systems

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An Ising chain is considered with a potential of the form $J(i, j)/|i-j|^{\alpha}$, where the $J(i, j)$ are independent random variables with mean zero. The chain contains both randomness and frustration, and serves to model a spin glass. A simple argument is provided to show that the system does not exhibit a phase transition at a positive temperature if $\alpha > 1$. This is to be contrasted with a ferromagnetic interaction which requires $\alpha > 2$. The basic idea is to prove that the "surface" *free* energy between two half-lines is finite, although the "surface" energy may be unbounded. For *d*-dimensional systems, it is shown that the free energy does not depend on the specific boundary conditions if $\alpha > (1/2)d$.

KEY WORDS: Phase transition; random interactions; long-range interactions; one-dimensional; relative entropy; free energy.

1. INTRODUCTION

One-dimensional Ising spin systems may exhibit a phase transition if the range of the interaction between the spins is long enough. To be specific, let us suppose that the spins interact via a potential of the form $J(i, j)/|i-j|^{\alpha}$. If all the $J(i, j)$ are equal and ferromagnetic, we have a phase transition[(1)] if $1 < \alpha \leqslant 2$, no phase transition[(2)] for $\alpha > 2$, and the free energy does not exist for $\alpha \leqslant 1$.

Suppose now that the $J(i, j)$ are independent random variables. If their mean is nonzero, one must require that $\alpha > 1$, as before, so as to obtain a finite free energy per site. However, in the theory of spin glasses one usually assumes that the mean of the $J(i, j)$ vanishes. In that case, the random

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variables $J(i, j)$ effectively *reduce*[2] the range of the interaction and α only has to exceed 1/2. That is, the minimal value of α above which thermodynamics exists and is nonrandom is reduced by a factor 1/2. This suggests that, instead of $\alpha > 2$, now $\alpha > 1$ suffices to make the equilibrium (Gibbs) state of an Ising chain unique and, hence, to exclude a phase transition. We will prove that this intuitive argument is indeed correct. More precisely, it will be shown that $\mu_\beta(S(i)) = 0$ whatever the temperature β and the equilibrium (Gibbs) state μ_β, provided $\alpha > 1$. Physically this means that the spin-flip symmetry is not broken and that the Edwards–Anderson order parameter vanishes. Partial results have been obtained by Khanin,[(5)] who proved the uniqueness of the Gibbs state for $\alpha > 3/2$. Recently, Kotliar *et al.*[(6)] predicted a phase transition for $1/2 < \alpha < 1$ and the absence of a phase transition for $\alpha > 1$.

In Section 2 we spell out some useful definitions and summarize the main arguments to prove that the free energy per spin $f(\beta)$ exists and is nonrandom. We also show that $f(\beta)$ does not depend on the specific boundary conditions. These results hold in any dimension d. Then, in Section 3, we specialize to $d = 1$, exploit the topology of the real line, and present a simple argument showing that the spin-flip symmetry is not broken for $\alpha > 1$. Physically, the idea is to prove that the "surface" free energy between left and right-lines is almost surely finite, in spite of the fact that the surface energy may be unbounded. This idea is implemented by some methods derived from a previous paper,[(4)] to be referred to as I, and the notion of *relative entropy*,[3] which is explained in Appendix A. A discussion of our results may be found in Section 4.

2. THE FREE ENERGY

For the moment we consider a random pair interaction with Hamiltonian

$$H_\Lambda(\{J\}) = \frac{1}{2} \sum_{i,j \in \Lambda} J(i, j) \, |i-j|^{-\alpha d} \, S(i) \, S(j) \tag{2.1}$$

where Λ is a finite domain in $\mathbb{Z}^d$. Boundary conditions have not been included yet. The $J(i, j)$ are independent random variables whose distribution only depends on $(i-j)$ and satisfies a uniformity condition which allows for a convergent cumulant expansion; cf. I Eq. (2.6). Throughout

[2] See Refs. 3 and 4. In Ref. 4 two annoying little misprints have escaped our attention. The two lower bounds for the summation in Eq. (2.16) are $n = 2$ and not 1 and ln 2, respectively.

[3] See Refs. 7 and 8. The use of a relative entropy to prove the absence of phase transitions was initiated in a more abstract, C^*-algebraic, setting by H. Araki, Ref. 7b.

this section we use $\langle \cdots \rangle$ to denote averaging with respect to the random configuration $\{J\}$.

For a particular configuration $\{J\}$ of the random variables the corresponding free energy per site in the thermodynamic limit is given by

$$f(\{J\}) = \lim_{\Lambda \to \infty} \frac{1}{|\Lambda|} F(\Lambda; \{J\}) = -kT \lim_{\Lambda \to \infty} \frac{1}{|\Lambda|} \ln Z_\Lambda(\{J\}) \qquad (2.2)$$

where $|\Lambda|$ is the number of sites in Λ,

$$Z_\Lambda(\{J\}) = \operatorname{Tr} \exp -\beta H_\Lambda(\{J\}) \qquad (2.3)$$

is the partition function for the Hamiltonian $H_\Lambda(\{J\})$, and $\beta = 1/kT$ is the inverse temperature, which we put equal to 1 throughout what follows. For Ising spins $[S(i) = \pm 1]$ the trace is a sum over all $2^{|\Lambda|}$ spin configurations.

As to the existence of the limit in (2.2) we note that for $\alpha > 1$ the limit exists in the sense of van Hove with probability one, is nonrandom and independent of the boundary conditions.[9] For $\alpha > 1/2$, the limit in (2.2) exists in the sense of Fisher with probability one and, again, is nonrandom. We now have to require that the mean of the $J(i, j)$ vanish. The first proof of this result has been given by Khanin and Sinai,[3] who invoked the theory of large deviations. Their paper only covers the case of Ising spins with free boundary conditions. At the end the authors remark that more general boundary conditions and more general, classical, spins could be handled in a similar way. In I (Section 2) we presented a different argument based on a subadditive ergodic theorem[10] and the observation that the free energy $F(\Lambda; \{J\})$ is *sub*additive in Λ,

$$F\left(\bigcup_{i=1}^{k} \Lambda_i\right) \leqslant \sum_{i=1}^{k} F(\Lambda_i), \qquad \Lambda_i \cap \Lambda_j = \phi \qquad \text{if} \quad i \neq j \qquad (2.4)$$

since the Hamiltonian (2.1) is quadratic in the spin operators. Our approach applies to arbitrary n-component classical and quantum models. However, we also assumed free boundary conditions, as in (2.1).

In this section we want to give an explicit proof that, for classical spin systems, the limit in (2.2) does not depend on the specific boundary conditions. For the sake of simplicity we assume Ising spins and take random variables $J(i, j)$ with variance one.

In taking the thermodynamic limit we fix a sequence $\Lambda_n \to \infty$ and specify a set of boundary conditions $\sigma = \{\sigma_n\}$, i.e., given Λ_n, we specify $S(j) = \sigma(j)$ for $j \notin \Lambda_n$. We then may write $(\Lambda_n = \Lambda)$

$$\begin{aligned} H_\Lambda^\sigma &= \frac{1}{2} \sum_{i,j \in \Lambda} J(i, j) |i-j|^{-\alpha d} S(i) S(j) + \sum_{\substack{i \in \Lambda \\ j \notin \Lambda}} J(i, j) |i-j|^{-\alpha d} S(i) \sigma(j) \\ &= H_\Lambda + W_\Lambda^\sigma \end{aligned} \qquad (2.5)$$

where W_Λ^σ is the interaction between Λ and its surroundings. If the free energy is independent of the boundary conditions σ, then certainly

$$\lim_{\Lambda\to\infty}\frac{1}{|\Lambda|}W_\Lambda^\sigma=0$$

with probability 1. In fact, we will prove a slightly stronger result; cf. (2.12).

We start by eliminating the $S(i)$ with i in Λ,

$$\begin{aligned} W_\Lambda^\sigma &= \sum_{i\in\Lambda}\left\{\sum_{j\notin\Lambda} J(i,j)|i-j|^{-\alpha d}\sigma(j)\right\}S(i) \\ &\equiv \sum_{i\in\Lambda} W_i^\sigma S(i) \end{aligned} \tag{2.6}$$

W_Λ^σ assumes its maximum (or minimum) if we choose $S(i)=\operatorname{sgn}(W_i^\sigma)$ [or $-\operatorname{sgn}(W_i^\sigma)$]. We therefore estimate

$$\begin{aligned} \left\langle\sum_{i\in\Lambda}|W_i^\sigma|\right\rangle &= \sum_{i\in\Lambda}\langle|W_i^\sigma|\rangle \\ &\leqslant \sum_{i\in\Lambda}\langle\{W_i^\sigma\}^2\rangle^{1/2} \\ &= \sum_{i\in\Lambda}\left\langle\left\{\sum_{j\notin\Lambda}J(i,j)|i-j|^{-\alpha d}\sigma(j)\right\}^2\right\rangle^{1/2} \\ &= \sum_{i\in\Lambda}\left\{\sum_{j\notin\Lambda}|i-j|^{-2\alpha d}\right\}^{1/2} \end{aligned} \tag{2.7}$$

Here we used the Cauchy–Schwarz inequality and exploited the fact that $|S(i)|\leqslant 1$ and the $J(i,j)$ are independent random variables with mean zero and variance one. Plainly, if i ranges through Λ and $\alpha>1/2$, the last term in (2.7) is uniformly bounded by a constant times $|\Lambda|$. In fact, since $\Lambda\to\infty$ in the sense of Fisher, this term divided by $|\Lambda|$ converges to zero, and thus

$$\lim_{\Lambda\to\infty}\frac{1}{|\Lambda|}\sum_{i\in\Lambda}\langle|W_i^\sigma|\rangle=0 \tag{2.8}$$

whenever $\alpha>1/2$. To see this more clearly, we take the dimension $d=1$, assume $\Lambda=[-|\Lambda|+1,0]$, and consider the interaction between Λ and the right half-line. Instead of (2.7) we then find, taking advantage of the O symbol,

$$\begin{aligned} \sum_{i=-|\Lambda|+1}^{0}\left\{\sum_{j=1}^{\infty}|i-j|^{-2\alpha}\right\}^{1/2} &= \sum_{i=-|\Lambda|+1}^{0}\{O(|i|^{1-2\alpha})\}^{1/2} \\ &= \sum_{i=-|\Lambda|+1}^{0}\{O(|i|^{1/2-\alpha})\}=O(|\Lambda|^{3/2-\alpha}) \end{aligned} \tag{2.9}$$

If we divide (2.9) by $|\Lambda|$, we are left with $O(|\Lambda|^{1/2-\alpha})$, and (2.8) is established.

To show, however, that $|\Lambda|^{-1}W_\Lambda^\sigma$ itself converges to zero with probability one, we note that W_Λ^σ is a sum of *independent*, though not identically distributed, random variables; cf. (2.6) and (2.8). For the sake of convenience we again suppose that $d=1$, and put

$$w_i = |W_i^\sigma| - \langle |W_i^\sigma| \rangle \tag{2.10}$$

We find, as in (2.7) and (2.9),

$$\sum_{i=-1}^{-\infty} i^{-2}\langle w_i^2 \rangle \leqslant \sum_{i=-1}^{-\infty} i^{-2}\langle \{W_i^\sigma\}^2 \rangle$$
$$\sim \sum_{i=1}^{\infty} i^{-2} i^{1-2\alpha} = \sum_{i=1}^{\infty} i^{-1-2\alpha} < \infty$$

Moreover, $\langle w_i \rangle = 0$. These two facts enable us to apply a strong law of large numbers[11] and to conclude that, as $\Lambda \to \infty$,

$$\frac{1}{|\Lambda|} \sum_{i \in \Lambda} w_i \to 0 \tag{2.11}$$

with probability one. Writing out (2.11) and using (2.8) we obtain our main result,

$$|\Lambda|^{-1} \sum_{i \in \Lambda} |W_i^\sigma| \to 0 \tag{2.12}$$

with probability one. It trivially follows that $|\Lambda|^{-1} W_\Lambda^\sigma \to 0$ also.

For free boundary conditions the existence of the free energy has already been established. Using (2.12) and the Bogoliubov–Peierls inequality[12] one now easily verifies that, as $\Lambda \to \infty$, the free energy exists, is a nonrandom number, and is independent of the specific boundary conditions, whatever the dimension.

Let us now return to $d=1$. The previous estimates enable us to study $\langle |W_\Lambda^\sigma| \rangle$ as $\Lambda \to \infty$, *without* dividing by $|\Lambda|$. Combining (2.7) and (2.9) we see that

$$\limsup_{\Lambda \to \infty} \langle |W_\Lambda^\sigma| \rangle < \infty \qquad \text{if} \quad \alpha > 3/2 \tag{2.13}$$

This result suggests that, given $\{J\}$, the *interaction energy between Λ and its surroundings is bounded uniformly in Λ and σ*. If true, we could invoke a result of Bricmont *et al.*[2] so as to conclude that the Gibbs state which is obtained in the thermodynamic limit is absolutely continuous with respect to the free boundary Gibbs state. Since every equilibrium state may be decomposed into ergodic (extremal) components[13] and two absolutely continuous, ergodic, components must be equal, we would have proven the uniqueness of the Gibbs state, i.e., the absence of a phase transition. Indeed, for $\alpha > 3/2$ this was done by Khanin,[5] who carefully estimated the interaction energy between different parts of the chain. As we have seen (cf. the Introduction and Ref. 6) this result is not expected to be optimal. We, therefore, need another method.

3. ABSENCE OF SYMMETRY BREAKING

Instead of calculating the interaction energy between two half-lines we determine their *free*-energy difference and show that this quantity is finite. More precisely, we divide the line into two half-lines, $\Lambda_1 = (-\infty, 0]$ and $\Lambda_2 = [1, \infty)$, so that $\mathbb{Z} = \Lambda_1 \cup \Lambda_2$, Let μ_β or simply μ be an equilibrium (Gibbs) state. Since we can decompose μ into ergodic components,[13] we may assume that μ itself is ergodic (i.e., extremal Gibbs). We write $\mu = \mu_+$ and associate with μ_+ a density matrix $\exp(-H_+)$ where

$$H_+ = H_1 + H_2 + W \tag{3.1}$$

W is the interaction energy between Λ_1 and Λ_2, and H_1 and H_2 refer to Λ_1 and Λ_2. The Hamiltonians in (3.1) are formal expressions but their meaning is (or will be) clear from the context. We now flip all the spins in Λ_2 and obtain a new state μ_- with Hamiltonian $H_- = H_1 + H_2 - W$. Since spin flipping is a symmetry transformation, the entropy does not change, but the energy does. Hence we find a free-energy difference

$$\Delta F = \mu_+(-2W) \equiv S(\mu_+ | \mu_-) \tag{3.2}$$

which is called the *relative entropy* (par abus de langage). We will show that this quantity is bounded with probability one. If so, μ_+ is absolutely continuous with respect to μ_- ($\mu_+ \ll \mu_-$), i.e., if $\mu_-(A) = 0$, then $\mu_+(A) = 0$, and we can find a density $g(\omega)$ so that $d\mu_+(\omega) = g(\omega)\, d\mu_-(\omega)$, where $g = d\mu_+/d\mu_-$ is the Radon–Nikodym derivative.[14] (See Appendix A.) For the moment we keep all this on ice and continue the argument.

If we flip *all* the spins, we transform μ into a spin-flipped state ν. Since μ is ergodic, so is ν. However, μ may be obtained as the (weak-*) limit of states μ_N where all spins *outside* $[-N, N]$ have been flipped ($N \to \infty$). That is, μ is a limit of states which are absolutely continuous with respect to ν and, hence, so is μ itself by the uniform estimate we just indicated.[4] But because μ and ν are ergodic they must be equal. So μ equals its spin-flipped state and we have $\mu(S(i))=0$, whatever i. Note that we did not average over the $J(i, j)$. Therefore, the spin-flip symmetry is not broken, the Edwards–Anderson order parameter vanishes, and a normal phase transition is to be excluded.

We now turn to the proofs. To simplify the notation we assume that all the $J(i, j)$ are Gaussian with mean zero and variance one. Gaussians have a particularly simple cumulant expansion. One may allow for more general distributions by using the methods of I, Eqs. (2.6) and (2.13)–(2.18). Moreover, let us define

$$P(H)=\ln \operatorname{tr} \exp(-H) \tag{3.3}$$

where tr (in contrast to Tr) is a *normalized* trace. Then $P(H)$ vanishes when $H=0$. Finally, let $\mathbb{E}\{\cdots\}$ denote an average over the $J(i, j)$ with $i\in \Lambda_1$ and $j\in \Lambda_2$, i.e., the random variables which occur in W. If ϕ is a non-negative function, which may depend on H_1 and H_2, and $\mathbb{E}\{\phi\}<\infty$, *uniformly* in H_1 and H_2, then ϕ is finite with probability one. This simple argument will be used repeatedly. Note that $\mathbb{E}\{\cdots\}$ does not refer to the random variables $J(i, j)$ in H_1 and H_2.

Before turning to our main theorem we prove some preparatory lemmas.

Lemma 1. Let

$$W=\sum_{\substack{i\in\Lambda_1\\ j\in\Lambda_2}} J(i, j)\,|i-j|^{-\alpha}\, S(i)\, S(j) \qquad \text{and} \qquad \alpha>1$$

Then $\mathbb{E}\{P(W)\}<\infty$.

Proof. By Jensen's inequality

$$\begin{aligned}\mathbb{E}\{\ln \operatorname{tr} \exp(-W)\} &\leqslant \ln \mathbb{E} \operatorname{tr} \prod_{\substack{i\in\Lambda_1\\ j\in\Lambda_2}} \exp\{-J(i, j)\,|i-j|^{-\alpha}\, S(i)\, S(j)\}\\ &=\ln \operatorname{tr} \prod_{\substack{i\leqslant 0\\ j>0}} \exp\{1/2\,|i-j|^{-2\alpha}\}<\infty\end{aligned}$$

since $\alpha>1$ and the trace is normalized. ∎

[4] This estimate involved μ_+ and μ_- and, thus, a translation-invariant (homogeneous) distribution (see Appendix B).

Lemma 2. For any H_1 and H_2, quadratic (even) in the spins, with H_1 defined on Λ_1 and H_2 on Λ_2,

$$\tilde{P}(W) \equiv P(H_1 + H_2 + W) - P(H_1 + H_2) \geqslant 0 \tag{3.4}$$

and

$$\mathbb{E}\{P(H_1 + H_2 + W) - P(H_1 + H_2)\} < \infty \tag{3.5}$$

Proof. The inequality (3.4) is nothing but $F(\Lambda_1 + \Lambda_2) \leqslant F(\Lambda_1) + F(\Lambda_2)$; compare (2.4) and note that $Z(H_1 + H_2) = Z(H_1)\, Z(H_2)$. The second inequality follows from

$$\begin{aligned}\mathbb{E}\left\{\ln \frac{Z(H_1 + H_2 + W)}{Z(H_1 + H_2)}\right\} &\leqslant \ln \mathbb{E}\left\{\frac{Z(H_1 + H_2 + W)}{Z(H_1 + H_2)}\right\} \\ &= \ln \mathbb{E}\{\mu_{H_1 + H_2}(e^{-W})\} = \ln \mu_{H_1 + H_2}(\mathbb{E}\{e^{-W}\}) \\ &= \ln \exp\left\{\frac{1}{2} \sum_{\substack{i \leqslant 0 \\ j > 0}} |i - j|^{-2\alpha}\right\} < \infty \end{aligned} \tag{3.6}$$

as in lemma 1. ▌

Physically, $-\tilde{P}(W)$ is the *surface free energy* which is obtained by coupling the two half-lines Λ_1 and Λ_2 by W. Combining (3.4) and (3.5) we find that this surface free energy is finite with probability one. We now want to make contact with the notion of *relative entropy*.

Lemma 3. Let $\{p_i\}$ and $\{q_i\}$ be two sets of positive numbers which both sum to one. Then

$$S = \sum_i p_i \{\ln p_i - \ln q_i\} \geqslant 0 \tag{3.7}$$

Proof. By Jensen's inequality

$$S = \sum_i q_i \left(\frac{p_i}{q_i}\right) \ln \left(\frac{p_i}{q_i}\right) \geqslant \left(\sum_i p_i\right) \ln \left(\sum_i p_i\right) = 0$$

since $x \ln x$ is convex for $x \geqslant 0$. ▌

We substitute $p_i = \exp[-(H_1 + H_2 + W)]/Z(H_1 + H_2 + W)$ and $q_i = \exp[-(H_1 + H_2 - W)]/Z(H_1 + H_2 - W)$ into (3.7) and sum instead of i over all spin configurations. Since $Z(H_1 + H_2 - W) = Z(H_1 + H_2 + W)$ we then find

$$0 \leqslant S = \sum -2W \frac{\exp[-(H_1 + H_2 + W)]}{Z(H_1 + H_2 + W)} = \mu_+(-2W) \tag{3.8}$$

Comparing (3.8) and (3.2) we see that $S = S(\mu_+ | \mu_-)$.

We rewrite S as a product of two terms,

$$S = \frac{2Z(H_1 + H_2)}{Z(H_1 + H_2 + W)} \cdot \mu_{H_1 + H_2}(-We^{-W}) \equiv (\mathrm{I}) \cdot (\mathrm{II}) \tag{3.9}$$

The second is positive by (3.8), and the first is finite with probability one or better. If H_1 and H_2 are even, this directly follows from the subadditivity of the free energy; cf. Eq. (2.4). In that case $Z(H_1 + H_2)/Z(H_1 + H_2 + W) \leqslant 1$. Alternatively we start by applying Jensen's inequality once again,

$$\frac{Z(H_1 + H_2 + W)}{Z(H_1 + H_2)} = \mu_{H_1 + H_2}[\exp(W)] \geqslant \exp[\mu_{H_1 + H_2}(W)]$$

so that

$$\mathbb{E}\left\{\frac{Z(H_1 + H_2)}{Z(H_1 + H_2 + W)}\right\} \leqslant \mathbb{E}\{\exp[-\mu_{H_1 + H_2}(W)]\}$$

$$= \exp\left\{\frac{1}{2} \sum_{\substack{i \leqslant 0 \\ j > 0}} |i - j|^{-2\alpha} \mu_{H_1 + H_2}[S(i)\, S(j)]^2\right\} < \infty$$

as in Lemma 1. Equation (3.9) exemplifies that there is a close relationship between the relative entropy (3.8) and the surface free energy (3.4). We are now prepared to prove the absence of symmetry breaking for $\alpha > 1$.

Theorem. In the thermodynamic limit, any Gibbs state μ of the random Ising chain does not break the spin-flip symmetry if $\alpha > 1$, i.e., $\mu(S(i)) = 0$ for all i.

Proof. It suffices to show that $\mathbb{E}\{\mu_{H_1 + H_2}[-W \exp(-W)]\} < \infty$ uniformly in H_1 and H_2. Then the expression between the curly brackets, (II) in (3.9), is finite for almost every random configuration, and the result is established.

By Fubini we may interchange $\mathbb{E}\{\cdots\}$ and the thermodynamic expectation with respect to H_1 and H_2. So it suffices to study

$$\mathbb{E}\{-W \exp(-W)\} = \sum_{\substack{i \leqslant 0 \\ j > 0}} \mathbb{E}\Bigg\{-J(i,j)\, |i - j|^{-\alpha} S(i)\, S(j)$$

$$\times \exp\Bigg[-\sum_{\substack{k \leqslant 0 \\ l > 0}} J(k,l)\, |k - l|^{-\alpha} S(k)\, S(l)\Bigg]\Bigg\} \tag{3.10}$$

Let us concentrate on a specific pair (i, j):

$$\mathbb{E}\{-J(i,j)|i-j|^{-\alpha} S(i) S(j) \exp[-J(i,j)|i-j|^{-\alpha} S(i) S(j)]\}$$
$$\times \mathbb{E}\left\{\exp -\left[\sum_{k,l}' J(k,l)|k-l|^{-\alpha} S(k) S(l)\right]\right\} \qquad (3.11)$$

where the primed sum does not contain the pair (i, j). The first factor in (3.11) is a Gaussian integral of the form

$$\int_{-\infty}^{+\infty} \frac{dx}{(2\pi)^{1/2}} e^{-(1/2)x^2} xt\, e^{xt} = t^2 e^{(1/2)t^2}$$

The second is even simpler. Collecting terms we get, for $\alpha > 1$,

$$(3.10) = \left(\sum_{\substack{i \leqslant 0 \\ j > 0}} |i-j|^{-2\alpha}\right) \exp\left(\frac{1}{2} \sum_{\substack{i \leqslant 0 \\ j > 0}} |i-j|^{-2\alpha}\right) < \infty \qquad (3.12)$$

as advertized. ▮

4. DISCUSSION

For a ferromagnetic Ising chain with long-range interactions of the form $|i-j|^{-\alpha}$ and $\alpha > 2$ there exists no phase transition in the sense that there is a unique Gibbs state[2] and the free energy $f(\beta)$ is analytic in β.[15] As was already pointed out in the Introduction, there is a phase transition for $1 < \alpha \leqslant 2$.[1] Heuristically this can be understood by noting that the surface energy between a left half-line and a right half-line is finite whenever $\alpha > 2$ and infinite for $1 < \alpha \leqslant 2$. As we have seen in Section 2, the bond randomness may effectively *de*crease the interaction. Here also it seems natural to find bounds for the surface energy W, and precisely this was done by Khanin.[5] In close analogy to the nonrandom case there is no phase transition in the sense that the Gibbs state is unique[5] and $f(\beta)$ is infinitely differentiable (C^∞) in β[16] whenever $\alpha > 3/2$. Our equation (2.13) is consistent with this result. We cannot exclude, however, that

$$\|W\|_\infty = \sup\left\{\sum_{\substack{i \in \Lambda_1 \\ j \in \Lambda_2}} J(i,j)|i-j|^{-\alpha} S(i) S(j)\right\} = +\infty \qquad (4.1)$$

for $1 < \alpha \leqslant 3/2$. Here the supremum is taken with respect to the spin configurations in two disjoint, neighboring half-lines Λ_1 and Λ_2. If we take Gaussian $J(i, j)$ and use (2.7) [with the *in*equality sign replaced by equality] and (2.9), we find that (4.1) indeed holds for $\alpha \leqslant 3/2$. But although for $1 < \alpha \leqslant 3/2$ the surface energy diverges, the surface *free* energy is always finite. We, therefore, have analyzed the latter quantity, in particular $\mu_{H_1+H_2}[-2W\exp(-W)]$, so as to prove the absence of symmetry breaking. Note, however, that the result directly follows once $\|W\|_\infty < \infty$ as in the ferromagnetic case[(2)] for $\alpha > 2$ and in the random case[(5)] for $\alpha > 3/2$.

If one wants to apply these results directly to real spin glasses, a caveat is in order. One has to realize that the spin-glass problem is a random-*site*, not a random-bond problem.[(17)] Nevertheless we think that it is satisfying to have some clear-cut results on a closely related issue, the existence or absence of phase transitions in highly frustrated systems with long-range interactions.

Note added: The arguments of this paper may be extended so as to cover the case of the two-dimensional XY model (van Enter and Fröhlich, manuscript in preparation).

APPENDIX A

In this Appendix we present an informal discussion of the notion of relative entropy. More in particular, we want to indicate why the boundedness of $S(\mu \mid \nu)$ implies that μ is absolutely continuous with respect to ν. For full mathematical rigor and further references the reader is referred to Fröhlich and Pfister.[(7a)]

If we are given two probability measures μ and ν on a phase space Ω, equation (3.7) suggests that $S(\mu \mid \nu)$ be given by

$$0 \leqslant S(\mu \mid \nu) = \int_\Omega d\mu \ln\left(\frac{d\mu}{d\nu}\right) \tag{A1}$$

where $d\mu/d\nu$ is a Radon–Nikodym derivative.[(14)] Let us denote by A^c the complement $\Omega - A$ of A. We first want to prove a useful inequality.

Lemma A. For any measurable A in Ω,

$$0 \leqslant \mu(A) \ln \frac{\mu(A)}{\nu(A)} + \mu(A^c) \ln \frac{\mu(A^c)}{\nu(A^c)} \leqslant S(\mu \mid \nu) \tag{A2}$$

Proof.

$$S(\mu \mid \nu) = \int_A d\nu \frac{d\mu}{d\nu} \ln\left(\frac{d\mu}{d\nu}\right) + \int_{A^c} d\nu \frac{d\mu}{d\nu} \ln\left(\frac{d\mu}{d\nu}\right)$$

Define normalized measures $\mu'(\cdot)=\mu(\cdot)/\mu(A)$ and $\nu'(\cdot)=\nu(\cdot)/\nu(A)$ for A and their analogs μ'' and ν'' for A^c. Then

$$S(\mu\,|\,\nu)=\mu(A)\int_A d\nu' \frac{d\mu'}{d\nu'}\ln\left(\frac{d\mu'}{d\nu'}\right)+\mu(A^c)\int_{A^c} d\nu'' \frac{d\mu''}{d\nu''}\ln\left(\frac{d\mu''}{d\nu''}\right)$$
$$+\left[\mu(A)\ln\frac{\mu(A)}{\nu(A)}+\mu(A^c)\ln\frac{\mu(A^c)}{\nu(A^c)}\right] \tag{A3}$$

Since Radon–Nikodym derivatives are positive and $x\ln x$ is convex for $x\geqslant 0$, the first two terms in (A3) are positive by Jensen's inequality, as in Lemma 3 (Section 3), and the inequality (A2) is established. ▌

Suppose now that $S(\mu\,|\,\nu)\leqslant K<\infty$ and that at the same time we can find an A in Ω such that $\nu(A)=0$ and $\mu(A)>0$. Then $\mu(A)\ln[\mu(A)/\nu(A)]=+\infty$ and, by Lemma A, this immediately leads to a contradiction. Hence $\nu(A)=0$ implies $\mu(A)=0$, i.e., μ is absolutely continuous with respect to ν if $S(\mu\,|\,\nu)$ is finite.

APPENDIX B

It suffices to show that $\Delta F_i\equiv f_i$, taken with respect to a site i, remains bounded as $i\to\infty$ (or $-\infty$) with probability one. In fact, we only need to prove this for a *sub*sequence.

Let Ω denote the probability space of all bond configurations $\{J_{ij}\}$ and let ω be an element of Ω, i.e., a specific bond configuration. We want to show that $f_i(\omega)\geqslant 0$ can be bounded by a constant, which may depend on ω, for some sequence $i\to\infty$ (or $-\infty$).

The f_i are positive, identically distributed but by no means independent random variables, with finite mean $\langle f_i\rangle=\langle f\rangle$, and $f_{i+1}(\omega)=f_i(T\omega)$ where T shifts by one. Let

$$F_N=\frac{1}{2N}\sum_{i=-N}^{N} f_i \tag{B1}$$

By the ergodic theorem,[18] $F_N(\omega)\to F^*(\omega)$ as $N\to\infty$ for almost every ω and $\langle F^*\rangle=\langle f\rangle$. Since F^* is positive, $F^*(\omega)<\infty$ with probability one. Plainly, as $N\to\infty$, we can find a subsequence such that $f_i(\omega)\leqslant F^*(\omega)$, as advertized.

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